Mathematical Modelling and Simulation of Tidal Energy

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Abstract: This paper presents a mathematical model to explain the phenomenon of tidal energy and the creation of a Matlab/Simulink to illustrate how the tidal current energy system works which is essential in power generation. A thorough explanation of tidal energy, including the science of tidal power, as well as the different operating mode has been discussed in this paper. Various literatures has been reviewed, the method used in this work were adopted from Sir Isaac Newton Famous Law of Gravitation. The derived model were applied in Matlab/Simulink, an output result were analyzed. Based on these result it was argued that tidal current has huge amount of energy that can be extracted and converted to a usable form. Tide can be predicted years in advance, its deployment has several benefits.

Keywords: Renewable energy, mathematical modelling, tidal energy phenomenon, MATLAB

1.0 INTRODUCTION

At present time there are great worries over climatic change globally, as well as creating awareness on the population worldwide about the need on minimizing greenhouse gases emissions. This has cause a focus on power generation from alternative sources such as tidal energy. Tidal power, also known as tidal energy is a kind of sustainable energy generated by the periodic change in the ocean envelope due to the interaction between the gravitational force of the moon and the sun on the earth and the centrifugal force resulting from the rotation of the earth and moon about each other [1]. The magnitude of the tide generating force due to their respective masses and distance from earth is about 32 percent sun and 68 percent moon. This shows that the gravitational force exerted by the moon on the earth is larger than that of the sun. It is about 2.125 times larger than the sun, due to the smaller distance between the earth and the moon [2]. Due to the effect of the gravitational force, a bulge of water is created being greater on the side of the earth nearest to the moon. Mutually, due to the centrifugal pull due to the rotation of the earth-moon system, another bulge of water is created, but here the bulge of water is created on the earth side farthest away from the moon. A resultant bulge is now created as a result of the two forces around the earth as it is shown in figure 1 and figure 2.

Figure 1: Rotation of Earth and Moon producing Tide
Source: [2]

Figure 2: Lunar tide created by the Earth – moon system

The tidal phenomenon occurs twice every 24h, 50min and 28s [3]. When the moon and the sun are in line whether they are pulling on the same side or on the opposite side (i.e full or new moon) the both gravitational effect join together causing high tides, known as spring tides. Similarly, when the moon and the sun are at 90°(at right angle) to each other, their gravitational force pull water to cancel each other resulting in low tide.
known as neap tides. The high tides are not very high and low tides are not very low during neap tides. Neap tides are experience at an interval of two weeks. But during spring tide the high tides are very high and low tides are very low. Spring tides are also experience in two week intervals. Neap tide usually occur a week after the spring tide [4]. Low tides and high tides are observed twice every 24hours and the hour (time) from low to hide tide is about six (6) hours [5].

In Nigeria, this phenomenon is much observable during the dry (winter) season which is around October to April particularly in the Southern region. However, during the rainy (summer) season around May to September it is observable at the coast line areas while other parts of the country experience little or no tide. The rain starts in the South and travels Northward with most part of the country seeing the most rain in May, June or July.

The periodicity of tidal phenomenon varies in consonance with the lunar and solar gravitational effects, respective movement of the moon and sun and other geographical bodies. The moon orbits the earth, every 29.53 days called lunar cycle (synodic month).

There are majorly three forms of tidal phenomena at different locations or places of the earth [6]. These are:

(a) **Semidiurnal tides**: This form of tide due to the earth rotation relative to both moon and sun has a period of about 12h 25min 14s, as a result the tidal phenomenon occurs every 24h 50min 28s, so each land mass are made to experience two low and two high tides during each period of rotation as illustrated in figure 3. The tide amplitude changes according to the lunar month with higher tidal range during new moon or full moon, when the moon and sun are inline. During half moon neap tides is experienced as the resultant gravitational pull is minimal. However, one of the tide has greater range than the other, having a higher high and lower low, therefore, a greater tidal flow while water is coming in and going out during the period between high and low level as shown in figure 3

(b) **Diurnal tides**: This tide consist of only one high and low tide a day. This type of tide is found in China sea and Tahiti. In this case the tidal period is of 24h 50min 28s, a full revolution of the moon around the earth. During each earth rotation, a point of the earth surface will pass through different path of equilibrium tide envelope and therefore experience a diurnal in tide levels

![Figure 4: Diurnal tides. Source: [7]](image)

(b) **Mixed tides**: This forms of tide joined the characteristic of diurnal and semidiurnal tides, thereby consisting of two uneven low and high tides each 24hours (a day). They are usually found in the Mediterranean sea and at Saigon.

![Figure 5: Mixed tides. Source: [8]](image)

This study helps to enhance our knowledge of the phenomenon of tidal energy. It enables us to make inform decision on tidal energy deployment. Its main benefits over other sustainable sources of energy is that it can be predicted years in advanced.
Tidal power deployment has some other benefits, these include, reduced greenhouse gas emission by utilizing tidal power in the place of fossil fuels. On tidal power barrages traffic or rail bridges can be built and these can enhance transport across estuaries. A clear understanding of tidal energy is also an essential factor for shipping, fishing and coast protection.

1.1 Tidal Energy Technologies

There are principally three technologies for tidal power generation, namely: tidal barrage, tidal stream and tidal lagoons. [3]

1.1.1 Tidal barrage: A tidal barrage is usually a dam, built across an estuary or a bay that experiences a tidal range in excess of 5m [2]. In tidal barrage the potential energy of the tides is used.

The principle is such that when the tide is high sluice gates are opened, which permit the flow of water from the ocean into a holding basin. This flowing water is allowed to pass through a turbine, thereby generating electrical power. The sluice gates are then closed at the peak of high tide and are reopened when the tide recedes allowing the water to flow through the turbine from the basin to the ocean thereby generating electrical power once again. From this principle researchers have mentioned different ways of extraction of energy like flood generation, pumping ebb generation, ebb and flood generation, two basin schemes e.t.c. A very good example of barrage method is La Rance tidal power plant, France [9]. The advantage of using barrage method to generate electricity in comparison with fossil fuel is that green house effect is reduce, thereby providing a better and conducive environment.

1.1.2 Tidal Stream: This technique makes use of the kinetic energy of the tide. Here a turbine is placed horizontally in the path of tidal current to generate electricity. This is almost the same with the operation of wind turbine. This approach is gaining popularity because of its reduced cost and ecological impact which is much minimal compare to barrage system. This technology has made massive progress towards commercialization in the last decade. Intensive research is being carried out in the UK waters related to tidal stream energy. By 2020 UK has a target of achieving 20 percent of its energy can be calculated [11, 12].

\[ E = \frac{1}{2} A \rho g h^2 \]  
(1)

Where,
- \( A \) = Barrage basin horizontal area
- \( \rho \) = Water density = 1025kg/m\(^3\) (seawater varies between 1021kg/m\(^3\) and 1030kg/m\(^3\))
- \( g \) = Acceleration due to the earth gravity (9.81m/s\(^2\))
- \( h \) = The vertical tidal range

From equation (1) it is observed that the tidal potential energy changes with square of tidal range. It is therefore suggested that a barrage should be positioned in a way that maximum storage head can be obtain. Black and Veatch [13] stipulate that the ideal water depth to obtain a better power output at few potential sites around the UK range is between 25m and 40m. Frost et al. [14] suggested that the recommended diameter of the rotor should range between 10m and 20m. The venturi effect produce as a result of the inlet accelerates the water as it is forced through a channel with a smaller cross-sectional area [15].

1.1.2.2 Kinetic Energy: The energy due to the tidal stream flowing across the cross section with a velocity is given by: [15]

\[ P = \frac{1}{2} c_p \rho A v^3 \]  
(2)

Where,
- \( c_p \) = The power coefficient
- \( \rho \) = The density of seawater (kg/m\(^3\))
- \( A \) = Channel cross sectional area (m\(^2\))
- \( v \) = The current velocity (ms\(^{-1}\))

The turbine design is always proportional to the power output or efficiency of the turbine\(^*\). The power output or efficiency of the turbine can be calculated using the equation below: [11]

\[ P = \frac{\varepsilon \rho A V^3}{2} \]  
(3)

Where,
- \( P \) = Power generated in (Watts)
- \( \varepsilon \) = Turbine efficiency or power output
- \( \rho \) = Water density, (seawater is 1025kg/m\(^3\))
- \( A \) = Turbine sweep area (m\(^2\))
- \( v \) = Velocity of the flow (m/s)
1.3 Tidal Energy around the Globe

Tidal energy is being regarded globally as a potential source of sustainable energy [16]. Tides at its high degree are found in many locations around the world. Some of these are: The strait of Alaska, the strait of Messina, the English Channel, Indonesia, the Aleutian, the fjords of Norway, the Pentland Firth, the Philippines, the seven estuary, Italy, Turkey, Columbia e.t.c. There are forecast that about 8,000MW can be extracted from the west coast of India, 20,000MW at the inlet of Mezen river and White sea and 90,000MW from off the North west coast of Russia. Research has also shown that there are about 106 locations in the European territorial waters for extracting tidal energy that would provide electricity of 48TW in a year [17]. Some regions have the potential to establish tidal power stations. Some of these regions are given in table 1. In these places tidal power plants have already been established while some are still in planning stage.

Table 1: Extreme tide of the world [12]

<table>
<thead>
<tr>
<th>Location</th>
<th>Country</th>
<th>Tidal Elevation (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bay of Mezen (White sea)</td>
<td>Russia</td>
<td>10</td>
</tr>
<tr>
<td>Puerto Gallegos</td>
<td>Argentina</td>
<td>13.3</td>
</tr>
<tr>
<td>Penzhinskaya Guba (Sea of Okhotsk)</td>
<td>Russia</td>
<td>13.4</td>
</tr>
<tr>
<td>La Rance</td>
<td>France</td>
<td>13.5</td>
</tr>
<tr>
<td>Severn Estuary</td>
<td>England</td>
<td>14.4</td>
</tr>
<tr>
<td>Port of Ganville</td>
<td>France</td>
<td>14.7</td>
</tr>
<tr>
<td>Bay of fundy</td>
<td>Canada</td>
<td>16.2</td>
</tr>
</tbody>
</table>

Table 2: Currently existing large tidal power plant [12]

<table>
<thead>
<tr>
<th>Site</th>
<th>Country</th>
<th>Bay Area Km²</th>
<th>Average Tide (m)</th>
<th>Installed Power (MW)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kislaya Guba</td>
<td>Russia</td>
<td>1.1</td>
<td>2.3</td>
<td>0.4</td>
</tr>
<tr>
<td>Jiangxia</td>
<td>China</td>
<td>1.4</td>
<td>5.08</td>
<td>3.9</td>
</tr>
<tr>
<td>Annapolis</td>
<td>Cannada</td>
<td>15</td>
<td>6.4</td>
<td>18</td>
</tr>
<tr>
<td>La Rance</td>
<td>France</td>
<td>22</td>
<td>8.55</td>
<td>240</td>
</tr>
</tbody>
</table>

2.0 METHODOLOGY

It was sir Isaac Newton, in his work “Philosophiae Naturalis, Principia Mathematics” [18] that first shows the connection between the gravitational force F, experienced between two bodies of mass M₁ and M₂ and the distance r, between them. He then put forward his universal famous law of gravitation, which is stated in equation form as [19]:

\[ F \propto \frac{M_1 M_2}{r^2} \]  

(4)

The magnitude of this resultant force can be written as:

\[ F = G \frac{M_1 M_2}{r^2} \]  

(5)

Where M₁ & M₂ are the two bodies or objects masses, r is the distance separating the bodies and G is a universal gravitational constant.

2.1 Tidal Generating Force

From Newton’s work, the tidal generating force is due to the gravitation of the sun and the moon on the earth. In order to explain this well, a system of two spherical bodies in form of a globe shall be considered, the Moon and the Earth, with their respective masses Mₑ and Mₘ. Considering a point O from the center of the Earth, to a point Q on the surface of the Earth, a distance vector \( \vec{r} \) is drawn and the corresponding distance vectors from O and Q to the Moon are denoted \( \vec{R} \) and \( \vec{d} \) respectively as soon in figure 6. These vectors length are written as \( r = ||\vec{r}||, R = ||\vec{R}|| \) and \( d = ||\vec{d}|| \). R and d can be regarded as a point, M since they are large compared to the radius of the Moon, and obviously we have:

\[ \vec{r} + \vec{d} = \vec{R} \]  

(6)

From Newton’s law of universal gravitation, the force of gravitation on the Earth is

\[ F_e = G \frac{M_m M_e}{R^2} \frac{\vec{R}}{R} \]  

(7)

Where G is the gravitational constant. The force gives the center of the Earth an acceleration:

\[ \vec{a}_o = G \frac{M_m}{R^2} \frac{\vec{R}}{R} = G \frac{M_m}{R^3} \vec{R} \]

Similarly, due to the gravitational pull of the Moon the acceleration at the point Q is

\[ \vec{a}_q = G \frac{M_e}{d^2} \frac{\vec{d}}{d} = G \frac{M_e}{d^3} \vec{d} \]

The difference between \( \vec{a}_o \) and \( \vec{a}_q \) is the tidal acceleration

\[ \vec{a}_o - \vec{a}_q = G M_m \frac{d - \vec{R}}{R^2} \]  

(8)

Which is in line with the tidal force per unit mass. The vector \( \vec{a}_o \) is contained in the plane through O, Q and M.
Using cosines rule, the relationship for triangle O, Q, M will be

\[ d^2 = R^2 + r^2 - 2Rr \cos \theta_m \]

Angle \( \theta_m \) is the angular zenith distance of the Moon. Therefore,

\[ d = R \left( 1 - \frac{2r}{R} \cos \theta_m \right) + 0 \left( \frac{r}{R} \right)^2 \]

By expressing the square-root in a series after the small parameter \( \frac{r}{R} \) and neglecting terms of order \( \left( \frac{r}{R} \right)^2 \), we shall have

\[ d \approx R \left( 1 - \frac{r}{R} \cos \theta_m \right) \]

Again by series expansion, we have

\[ \frac{1}{d^3} = \frac{1}{R^3 \left( 1 - \frac{r}{R} \cos \theta_m \right)^3} \approx \frac{1}{R^3} \left( 1 + \frac{3r}{R} \cos \theta_m \right) \]

By expressing the latter relation of eqn (8) together with eqn (6), the new tidal acceleration, eqn (8), becomes:

\[ \ddot{a} = g \frac{M_m}{R^2} \left[ 3 \frac{r}{R} \cos \theta_m - \frac{2}{r} \right] \]  

(9)

When introducing acceleration due to gravity at the earth surface,

\[ g = G \frac{M_e}{R^2} \]

Equation (9) can be re-written as:

\[ \ddot{a} = g \frac{M_m}{M_e} \left( \frac{r}{R} \right)^3 \left[ 3 \frac{r}{R} \cos \theta_m - \frac{2}{r} \right] \]  

(10)

From here we can also observed that the tidal acceleration is a very small fraction of g. Substituting the Constants for the Earth Moon system, we shall have \( \frac{M_m}{M_e} = 0.012 \), \( \frac{r}{R} = 0.017 \). The fraction is of the order \( 10^{-8} \). Equation (10), reveals that \( \ddot{a} \) is a vertical vector sum always pointing towards down the vertical and a vector toward the moon and \( \dot{R} \) direction for \( \theta_m < \frac{\pi}{2} \) and opposite \( \dot{R} \) when \( \frac{\pi}{2} < \theta_m < \pi \).

Therefore a component of the acceleration will be directed either towards the point X under the Moon or towards the corresponding point Y on the other side of the Earth as shown in figure 7.

Figure 7: Tidal acceleration direction.

The vector \( \ddot{a} \) can be disintegrated into an horizontal component, \( a_h \), and a vertical component, \( a_r \). The former is directed along the great circle arch XOY.

Since we have \( \dot{R} \times \ddot{r} = R \dot{r} \cos \theta_m \)

\[ a_r = \ddot{a} \frac{r}{R} = g \frac{M_m}{M_e} \left( \frac{r}{R} \right)^3 \left[ 3 \cos^2 \theta_m - 1 \right] \]

(11)

And since \( \dot{R} \times \ddot{r} = R \dot{r} \sin \theta_m \)

\[ a_h = \ddot{a} \frac{r}{R} = \frac{3}{2} g \frac{M_m}{M_e} \left( \frac{r}{R} \right)^3 \left[ 3 \sin 2\theta_m \right] \]  

(12)

When we introduce the moon horizontal parallax defined by

\[ \sin \pi_m = \frac{r}{R} \]

It is a generally used parameter for the position of the moon.

Now if we imagine that the earth is under the shade of a thin sheet of water subject to the tidal generating force of the moon. In order to attain equilibrium and to set up an adverse pressure gradient or difference annihilating the horizontal tidal force the surface of the water will deform. The equilibrium condition will be

\[ -g \frac{\delta \eta_m}{\rho \delta \theta_m} - \frac{3}{2} g \frac{M_m}{M_e} \sin 3\pi_m \cos 2\theta_m = 0 \]  

(13)

Where \( \eta_m \) is the displacement of the water vertically. By integrating eqn (13) we have

\[ \eta_m = \frac{3}{4} \frac{M_m}{M_e} r \sin 3\pi_m \cos 2\theta_m + C \]

Where \( C \) is a constant of integration. It is determined on the ground that the distortion does not cause a change in the water volume.

Substituting the constant of integration \( C = \frac{1}{3} \). Then \( \eta_m \) shall be

\[ \eta_m = \frac{1}{4} \frac{M_m}{M_e} r \sin 3\pi_m (3\cos 2\theta_m + 1) \]  

(14)
This expression equation (14) reveals that there will be high water under the moon at the point X and point Y on the other side of the earth. X region of low water will extend around the earth with lowest water level for \( \theta_m = \frac{\pi}{2} \) as illustrated in figure 8.

Figure 8: The equilibrium tide

Equation (14), an expression for the equilibrium tide is the resultant effect of the action of the moon.

Similarly, the resultant effect of the action of the sun will give rise to equation (15).

\[
\eta_s = \frac{M_s}{M_e} r \sin^3 \pi_s (3 \cos 2\theta_s + 1) \tag{15}
\]

Where \( M_s = \) The mass of the sun
\( \theta_s = \) The zenith distance of the Sun
\( \pi_s = \) The horizontal parallax of the Sun

However, the mass of the Moon is smaller than the mass of the Sun but the parallax of the Sun is less than the parallax of the Moon. The equilibrium tide due to the action of the Sun is therefore about half of the Moon. The resultant effect of the sun and the moon gives an expression for the total equilibrium tide

\[
\eta = \eta_m + \eta_s \tag{16}
\]

Substituting the constants in table 3, we have \( |\eta_m| \) as 0.35m and \( |\eta_s| \) as 0.15 m.
Therefore, the resultant equilibrium tide from the action of the sun and the moon is 0.50 m.

Table 3: Astronomical Constants.

<table>
<thead>
<tr>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass of the Sun: ( M_s )</td>
<td>1.991 \times 10^{27} kg</td>
</tr>
<tr>
<td>Mass of the Earth: ( M_e )</td>
<td>5.974 \times 10^{24} kg</td>
</tr>
<tr>
<td>Mass of the Moon: ( M_m )</td>
<td>7.347 \times 10^{22} kg</td>
</tr>
<tr>
<td>Earth-moon mean distance ( R )</td>
<td>3.844 \times 10^8 km</td>
</tr>
<tr>
<td>Earth-Sun mean distance ( R )</td>
<td>1.496 \times 10^9 km</td>
</tr>
<tr>
<td>Radius of the Earth ( r )</td>
<td>6.370 \times 10^3 km</td>
</tr>
</tbody>
</table>

2.2 Power Equation

Generating electricity from flowing water can be done either by constructing a tidal barrage across a bay in an area with high tide (tidal potential energy), or by obtaining energy through free flowing water (tidal kinetic energy). The amount of power that a tidal current turbine can extract from flowing water depends on the turbine design. Factors such as rotor diameter and tidal current speed affect this amount of power.

Power available in tidal currents is given by [19]:

\[
P_{\text{act}} = 0.5 \rho A V^3 \tag{17}
\]

Where,
\( \rho = \) Seawater density (1025 Kgm\(^{-3}\))
\( A = \) Rotor blade area (\( m^2 \))
\( V = \) Water current speed or velocity (m/s)

But it is important to note that a turbine or marine energy converter can only extract a portion of this power due to losses. so equation (16) can be modified as

\[
P_{\text{act}} = 0.5 \rho C_p A V^3 \tag{18}
\]

Where \( C_p = \) The power coefficient and the percentage of power that can be harness from the fluid stream taking into consideration losses due to internal mechanisms within the power converter or turbine and Betz’s law. For wind generators or turbine, the value of \( C_p \) ranges from 0.25 – 0.3 but for marine turbine, the value of \( C_p \) ranges from 0.35 – 0.5 [5].

Marwa et al [20] stipulate that an OpenHydro turbine having specifications in table 4, with 10 m rotor diameter, 2.57 m/s water speed can generate 1 MW rated power.

Table 4: Open Hydro Specifications

<table>
<thead>
<tr>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rotor Diameter</td>
<td>15 m</td>
</tr>
<tr>
<td>Rated Power</td>
<td>1.5 MW</td>
</tr>
<tr>
<td>Cut in Speed</td>
<td>0.7 m/s</td>
</tr>
<tr>
<td>Output Power</td>
<td>11KV AC, 50-60 HZ 3Φ</td>
</tr>
</tbody>
</table>

When the tidal current speed exceeds the 2.57 m/s speed range the extracted power will be limited to 1.5MW by power control strategy

3.0 RESULT AND DISCUSSION

We have seen that through the movement of the Moon in the equatorial plane there will be high water for each location or region on the Earth, also when the Moon passes the meridian, about 12 hours 25min 14sec later when the Moon is on the opposite side of the Earth, there will be another similarly high water.

Therefore, equation (14) shows the semi-diurnal behaviour of sea level. Also, there will be an asymmetry between two consecutive high tides when the Moon has a Northern or Southern declination. Hence we can say that equation (14) also shows the diurnal characteristics of tide. The surface displacement given by equation (14) is called the equilibrium tide. The equilibrium tide explains the resultant effect of the action of the sun and the moon on the earth. The water mass adjust simultaneously with the movement of the Moon. When the earth rotates it is thought to follow the Moon.

Figure 9 affirms the fact that high tides and low tides are experienced twice every 24hour and the time from high to low tide is about six (6) hour. It is clear from figure 10 that during one lunar month there are two spring tides (at times of new and full moon and two neap tides (at times of first and last quarter of the moon). This analogy repeats for all year months.

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**CONCLUSION**

Tidal power is a clear and sure renewable source of energy. It can be predicted years in advance. Its deployment can bring several benefits and developments to any nation. On its deployment the dam can acts as protection against floods and permitting links between both sides of the dam, this link can act
as a bridge, allowing transportation and other locomotion easier. Furthermore, its deployment can bring improvement in aquacultures and other marine life. This work has clearly explained mathematically the phenomenon and operation of tidal energy. MATLAB R2015a was used for the simulation. More so, it has some environmental effects as it could result to changes on the estuary ecological community. It can have negative effects on migratory species as well as gathering of silt or debris behind the barrage. But with a very careful barrage design and construction and with regular maintenance strategies, silt accumulation will not be a problem.

REFERENCE


