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Some Applications of Eigenvalues in Engineering Problems

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Abstract: In this paper, some applications of eigenvalues, which are useful in solving engineering problems, are presented.

Keywords: Mixing problem, Vibrating system.

1. INTORDUCTION

Eigenvalue problems are of practical interest to the engineers, physicists, and mathematicians; we shall see that their theory makes up a paper in linear algebra that has found numerous applications.In this paper we discuss a few typical examples from the range of applications of matrix eigenvalue problems. Matrix eigenvalue problems concern the solutions of the vector equations

$$\mathbf{Ax} = \lambda\mathbf{x}$$

where $\mathbf{A}$ is a given square matrix and vector $\mathbf{x} \neq \mathbf{0}$ and scalar λ are unknown. Here $\mathbf{x}$ is called the eigenvector of $\mathbf{A}$ and λ's are called the eigenvalues of $\mathbf{A}$.

2. EIGENVALUE

The eigenvalues of a square matrix $\mathbf{A}$ are the roots of the characteristic equation of $\mathbf{A}$. Hence an $n \times n$ matrix $\mathbf{A}$ has at least one eigenvalue and at most n numerically different eigenvalues.

2.1 Mixing Problem Involving Two Tanks

We consider mixing problem involving two tanks as follows.

Suppose that two tanks are connected as shown in the accompanying figure (1). At time $t = 0$, tank 1 contains 30 L of water in which 2 kg of salt has been dissolved, and tank 2 contains 30 L of water in which 5 kg of salt has been dissolved. Pure water is pumped into tank 1 at the rate of 20 L/min; the saline mixtures are exchanged between the two tanks at the rates shown, the mixture in tank 2 drains out at the rate of 15 L/min, and the mixture in tank 1 drains out at the rate of 5 L/min. We want to find the amount of salt in the two tanks at time t.

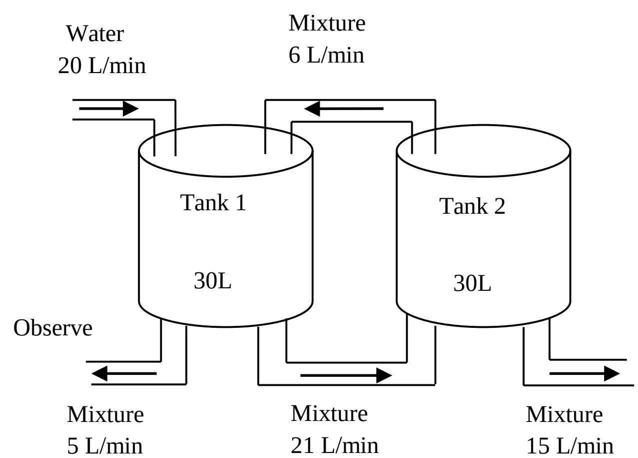

Figure 1. Mixing problem involving two

first that
the amount of liquid in each tank remains constant because the rate at which liquid enters each tank is the same as the rate at which it leaves. Thus each tank always contains 30L of liquid.

Now let

$y_1(t) =$ amount of salt in tank 1 at time t ,

$y_2(t) =$ amount of salt in tank 2 at time t ,

and let the rates of change be y_1' and y_2' .

Since the rate at which salt enters

$$\text{tank 1} = \left(6\frac{\text{L}}{\text{min}}\right)\left(\frac{y_2(t)}{30}\frac{\text{kg}}{\text{L}}\right) = \frac{y_2(t)}{5}\frac{\text{kg}}{\text{min}} \text{ and}$$

the rate at which salt leaves

$$\text{tank 1} = \left(26\frac{\text{L}}{\text{min}}\right)\left(\frac{y_1(t)}{30}\frac{\text{kg}}{\text{L}}\right) = \frac{13y_1(t)}{15}\frac{\text{kg}}{\text{min}},$$

the rate of change of $y_1(t)$ is

$$y_1'(t) = \text{rate in} - \text{rate out} = \frac{y_2(t)}{5} - \frac{13y_1(t)}{15}$$

Since the rate at which salt enters

tank 2 $= \left(21\frac{\mathrm{L}}{\mathrm{min}}\right)\left(\frac{y_1(t)}{30}\frac{\mathrm{kg}}{\mathrm{L}}\right) = \frac{7y_1(t)}{10}\frac{\mathrm{kg}}{\mathrm{min}}$ and

the rate at which salt leaves

tank 2 $= \left(21\frac{\mathrm{L}}{\mathrm{min}}\right)\left(\frac{y_2(t)}{30}\frac{\mathrm{kg}}{\mathrm{L}}\right) = \frac{7y_2(t)}{10}\frac{\mathrm{kg}}{\mathrm{min}}$,

the rate of change of $y_2(t)$ is

$y_2'(t) = \text{rate in} - \text{rate out} = \frac{7y_1(t)}{10} - \frac{7y_2(t)}{10}$

Mixture problem is the system of the first-order ODEs

$$y_1' = -\frac{13}{15}y_1 + \frac{1}{5}y_2$$

(Tank 1)

$$y_2' = \frac{7}{10}y_1 - \frac{7}{10}y_2$$

(Tank 2)

or

$$\mathbf{y}' = \mathbf{A}\mathbf{y} \quad \text{where}$$

$$\mathbf{A} = \begin{bmatrix} -\frac{13}{15} & \frac{1}{5} \\ \frac{7}{10} & -\frac{7}{10} \end{bmatrix} \text{ and } \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Let $\mathbf{y} = \mathbf{x}e^{\lambda t}$ be solution of $\mathbf{y}' = \mathbf{A}\mathbf{y}$ (1)

Then $\lambda\mathbf{x}e^{\lambda t} = \mathbf{A}\mathbf{x}e^{\lambda t}$

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \text{ where } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

which is an eigenvalue problem.

$$-\frac{13}{15}x_1 + \frac{1}{5}x_2 = \lambda x_1$$

$$\frac{7}{10}x_1 - \frac{7}{10}x_2 = \lambda x_2$$

$$\left(-\frac{13}{15} - \lambda\right)x_1 + \frac{1}{5}x_2 = 0$$

$$\frac{7}{10}x_1 + \left(-\frac{7}{10} - \lambda\right)x_2 = 0$$

$$(\mathbf{A} - \lambda \mathrm{I})\mathbf{x} = 0$$

For non trivial solutions,

$$\det(\mathbf{A} - \lambda \mathrm{I}) = \begin{vmatrix} -\frac{13}{15} - \lambda & \frac{1}{5} \\ \frac{7}{10} & -\frac{7}{10} - \lambda \end{vmatrix} = 0$$

$$\left(-\frac{13}{15} - \lambda\right)\left(-\frac{7}{10} - \lambda\right) - \frac{7}{50} = 0$$

$$\lambda^2 + \frac{47}{30}\lambda + \frac{7}{15} = 0 \qquad (2)$$

The solutions of quadratic equation are $\lambda_1 = -\frac{2}{5}$ and $\lambda_2 = -\frac{7}{6}$.

$\lambda_1 = -\frac{2}{5}$ corresponding equation are

$$-\frac{7}{15}x_1 + \frac{1}{5}x_2 = 0$$
$$\frac{7}{10}x_1 - \frac{3}{10}x_2 = 0$$

From these two equations, we have $x_2 = \frac{7}{3}x_1$

∴ we may choose $x_1 = 3$, then $x_2 = 7$.

$$\mathbf{x_1} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

$\lambda_2 = -\frac{7}{6}$ corresponding equation are

$$\frac{3}{10}x_1 + \frac{1}{5}x_2 = 0$$

$$\frac{7}{10}x_1 + \frac{7}{15}x_2 = 0$$

From these two equations, we have $x_2 = -\frac{3}{2}x_1$

∴ we may choose $x_1 = -2$, then $x_2 = 3$.

$$\mathbf{x_2} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

From (1)

$$y = c_1\mathbf{x_1}e^{\lambda_1 t} + c_2\mathbf{x_2}e^{\lambda_2 t} = c_1\begin{bmatrix} 3 \\ 7 \end{bmatrix}e^{-\frac{2}{5}t} + \begin{bmatrix} -2 \\ 3 \end{bmatrix}e^{-\frac{7}{6}t} \qquad (3)$$

where c_1 and c_1 are arbitrary constants.

Initial condition are $y_1(0) = 2$ and $y_2(0) = 5$.

Therefore,

$$y(0) = c_1\begin{bmatrix} 3 \\ 7 \end{bmatrix} + c_2\begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3c_1 - 2c_2 \\ 7c_1 + 3c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

Then $3c_1 - 2c_2 = 2$ and $7c_1 + 3c_2 = 5$

and hence $c_1 = \frac{16}{23}$, $c_2 = \frac{1}{23}$

$$y = \frac{16}{23}\mathbf{x_1}e^{-\frac{2}{5}t} + \frac{1}{23}\mathbf{x_2}e^{-\frac{7}{6}t} = \frac{16}{23}\begin{bmatrix} 3 \\ 7 \end{bmatrix}e^{-\frac{2}{5}t} + \frac{1}{23}\begin{bmatrix} -2 \\ 3 \end{bmatrix}e^{-\frac{7}{6}t}$$

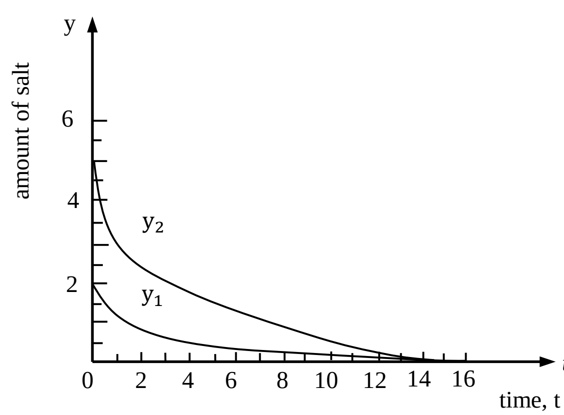

Figure 2. Leaving Salt Vs time graph

In components,

$y_1 = 2.096e^{-\frac{2}{5}t} - 0.09e^{-\frac{7}{6}t}$ (Tank 1, lower curve)

$y_2 = 4.87e^{-\frac{2}{5}t} + 0.13e^{-\frac{7}{6}t}$ (Tank 2, upper curve)

2.2 Vibrating System of Two Masses on Two Springs

Mass-spring systems involving several masses and springs can be treated as eigenvalue problems. The mechanical system in figure (3) is governed by the system of ODEs

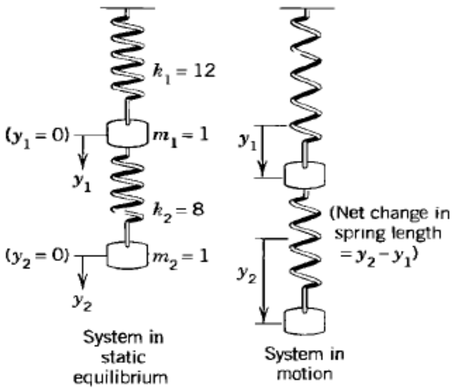

Figure 3. Masses on springs

$$y_1'' = -20y_1 + 8y_2 \qquad (1)$$

$$y_2'' = 8y_1 - 8y_2$$

where y_1 and y_2 are the displacements of masses from rest, as shown in figure (3), y_1 and y_2 primes denote derivatives with respect to time t.

$$y'' = \begin{bmatrix} y_1'' \\ y_2'' \end{bmatrix} = \mathbf{A}y = \begin{bmatrix} -20 & 8 \\ 8 & -8 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \qquad (2)$$

We try a vector solution of the form

$$y = \mathbf{x}\, e^{\omega t} \qquad (3)$$

$$y' = \omega \mathbf{x}\, e^{\omega t}$$

$$y'' = \omega^2 \mathbf{x}\, e^{\omega t}$$

Substitution into (2) $\omega^2 \mathbf{x}\, e^{\omega t} = \mathbf{A}\, \mathbf{x}\, e^{\omega t}$

$$\mathbf{A}\,\mathbf{x} = \lambda\,\mathbf{x} \qquad \text{where } \lambda = \omega^2 \qquad (4)$$

$$\mathbf{A} = \begin{bmatrix} -20 & 8 \\ 8 & -8 \end{bmatrix}$$

$$\mathbf{A}\,\mathbf{x} = \begin{bmatrix} -20 & 8 \\ 8 & -8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

In components,

$$-20x_1 + 8x_2 = \lambda x_1$$

$$8x_1 - 8x_2 = \lambda x_2$$

Transferring the terms on the right to the left, we get

$$(-20 - \lambda)x_1 + 8x_2 = 0$$

$$8x_1 + (-8 - \lambda)x_2 = 0 \qquad (5)$$

$$(\mathbf{A} - \lambda I)\mathbf{x} = 0$$

$$\det(\mathbf{A} - \lambda I) = \begin{vmatrix} -20 - \lambda & 8 \\ 8 & -8 - \lambda \end{vmatrix} = 0$$

$$(-20 - \lambda)(-8 - \lambda) - 64 = 0$$

$$\lambda^2 + 28\lambda + 96 = 0$$

The solutions of quadratic equation are $\lambda_1 = -4$ and $\lambda_2 = -24$.

Consequently, $\omega_1 = \pm 2i$ and $\omega_2 = \pm\, 2\sqrt{6}\, i$. Corresponding eigenvectors are

$$\mathbf{x_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{x_2} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \qquad (6)$$

From (3) we thus obtain

$$\mathbf{x_1} e^{\pm 2it} = \mathbf{x_1}(\cos 2t \pm i \sin 2t)$$

$$\mathbf{x_2} e^{\pm 2\sqrt{6}it} = \mathbf{x_2}(\cos 2\sqrt{6}\, t \pm i \sin 2\sqrt{6}\, t)$$

By addition and subtraction, we get

$$\mathbf{x_1} \cos 2t,\ \ \mathbf{x_1} \sin 2t,\ \ \mathbf{x_2}\ \cos 2\sqrt{6}\, t\ ,\ \ \mathbf{x_2} \sin 2\sqrt{6}\, t$$

A general solution is

$y = \mathbf{x_1}(a_1 \cos 2t + b_1 \sin 2t) + \mathbf{x_2}(a_2 \cos 2\sqrt{6}\, t + b_2 \sin 2\sqrt{6}\, t)$ with arbitrary constants a_1, b_1, a_2, b_2.

By (6) $y_1 = a_1 \cos 2t + b_1 \sin 2t + 2a_2 \cos 2\sqrt{6}\, t + 2b_2 \sin 2\sqrt{6}\, t$

$$y_2 = 2a_1 \cos 2t + 2b_1 \sin 2t - a_2 \cos 2\sqrt{6}\, t - b_2 \sin 2\sqrt{6}\, t$$

3. CONCLUSION

We had presented the usage of eigenvalues and eigenvectors to solve some engineering problems. There are many applications of eigenvalues and eigenvectors in various fields of engineering and sciences which will be discussed later.

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