International Journal of Science and Engineering Applications
Volume 8–Issue 08,390-394, 2019, ISSN:-2319–7560

Introduction to Interpolating Polynomial Curves

Kyaw Kyaw Than
Department of Engineering Mathematics
Technological University Thanlyin, Myanmar

Aye Aye Khine
Department of Mechnical Engineering
Yangon Technological University
Yangon, Myanmar

Wah Wah Aung
Department of Engineering Mathematics
Technological University Thanlyin, Myanmar

Abstract: This paper mentions the convex combinations and the convex hull of a set of points, and a brief and very informal introduction to parametric curves. The first method for curve construction, namely polynomial interpolation, is introduced in this paper.

Keywords: Polynomial interpolation, Cubic interpolating curves.

1. INTRODUCTION

Interpolation polynomials and Interpolating polynomial curves are fundamental tool for "Spline Interpolation".

2. CONVEX COMBINATION AND CONVEXHULL

2.1 Convex Combination

A convex combination is described as

$$c = (1-\lambda)c_1 + \lambda c_2 \qquad (1.1)$$

c is said to be a convex combination of c_1 and c_2.

Here, c_1 and c_2 are two given numbers and λa given weight in the range$[0,1]$. The result of the computation is the numbercwhich mustlie betweenc$_1$ and c_2 as average always do.

2.2 The Convex Hull of a Set of Points

If $c_1 = (x_1, y_1)$and $c_2 = (x_2, y_2)$, then

(1.1), is usually implemented on a computer by expressing it in terms of convex combinations of real numbers

$$(x,y) = \big((1-\lambda)x_1 + \lambda x_2, (1-\lambda)y_1 + \lambda y_2\big), \qquad (1.2)$$

where $(x,y) = c$ and the weight λ is some number in the range$0 \leq \lambda \leq 1$.

Asλ are all real numbers, the pointc in (1.2) will pass through c_1 and c_2 as a whole straight line.

The convex hull, or the set of all weighted averages, of the two pointsc$_1$ and c_2 is the part of the line between these two points when $0 \leq \lambda \leq 1$.

$c = \lambda_1 c_1 + \lambda_2 c_2 + \cdots + \lambda_n c_n$ is a convex combination of n points $(c_i)_{i=2}^{n}$

where$\sum_{i=1}^{n} \lambda_i = 1$,and $0 \leq \lambda_i$ for $i = 1,2,\ldots,n$.

The convex hull of two points is the straight line segment that connects the points, and the convex hull of three points can be identified with the triangle spanned by the points. In general, the convex hull of n points is the n-sided polygon with the points as corners.

3.POLYNOMIAL INTERPOLATION

3.1 Some Fundamental Concepts

The most natural curve to construct from given two points $c_0 = (x_0, y_0)$and $c_1 = (x_1, y_1)$is the straight line segment which connects the two points.

Generally, this line segment is expressed as

$$q(t|c_0, c_1; t_0, t_1) = \frac{t_1 - t}{t_1 - t_0} c_0 + \frac{t - t_0}{t_1 - t_0} c_1 \quad , \text{ for } t \in [t_0, t_1] \qquad (1.3)$$

Here, $t_0$and $t_1$are two arbitrary real numbers with $t_0 < t_1$.

The two coefficients$\frac{t_1 - t}{t_1 - t_0}$,$\frac{t - t_0}{t_1 - t_0}$add to one and each of them is non-negative as long ast is in the interval$[t_0, t_1]$.

Equation (1.3) is an example of a parametric representation.

If the variablet is denoted as time, the parametric representation $q(t|c_0, c_1; t_0, t_1)$ gives a way to travel from c_0 to c_1. The parameter t_0 is described as the time at starting point c_0 and t_1 as the time at end point c_1 .

The speed of travel along the curve is given by the tangent vector or derivative $q'(t|c_0, c_1; t_0, t_1) = \frac{|c_1 - c_0|}{t_1 - t_0}$,

while the scalar speed or velocity is given by the length of the tangent vector$|q'(t|c_0, c_1; t_0, t_1)| = \frac{|c_1 - c_0|}{t_1 - t_0} \frac{\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}}{t_1 - t_0}$.

3.2 Quadratic Interpolation of Three Points

The curve is constructed from given three interpolation points c_0, c_1 andc_2, which is quadratic and also needs three parameters$(t_i)_{i=0}^{2}$. First, the two straight lines are described as$q_{0,1}(t) = q(t|c_0, c_1; t_0, t_1)$ and $q_{1,2}(t) = q(t|c_1, c_2; t_1, t_2)$

Then, the quadratic curve is

$$q_{0,2}(t) = q(t|c_0, c_1, c_2; t_0, t_1, t_2)$$

$$= \frac{t_2-t}{t_2-t_0} q_{0,1}(t) + \frac{t-t_0}{t_2-t_0} q_{1,1}(t) (1.4)$$

At$t = t_0$,

$$q_{0,1}(t_0) = \frac{t_1 - t_0}{t_1 - t_0} c_0 + \frac{t_0 - t_0}{t_1 - t_0} c_1 = c_0$$

$$q_{0,2}(t_0) = \frac{t_2 - t_0}{t_2 - t_0} q_{0,1}(t_0) + \frac{t_0 - t_0}{t_2 - t_0} q_{1,1}(t_0) = q_{0,1}(t_0) = c_0$$

Thus, $q_{0,2}(t_0) = q_{0,1}(t_0) = c_0$

At$t = t_1$,

$$q_{0,1}(t_1) = \frac{t_1 - t_1}{t_1 - t_0} c_0 + \frac{t_1 - t_0}{t_1 - t_0} c_1 = c_1$$

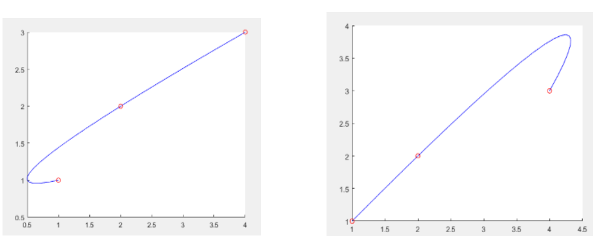
$$q_{1,1}(t_1) = \frac{t_2 - t_1}{t_2 - t_1} c_1 + \frac{t_1 - t_1}{t_2 - t_1} c_2 = c_1$$

$$q_{0,2}(t_1) = \frac{t_2 - t_1}{t_2 - t_0} q_{0,1}(t_1) + \frac{t_1 - t_0}{t_2 - t_0} q_{1,1}(t_1)$$

$$= \frac{t_2 - t_1}{t_2 - t_0} c_1 + \frac{t_1 - t_0}{t_2 - t_0} c_1 = c_1$$

At$t = t_2$,$q_{1,1}(t_2) = \frac{t_2-t_2}{t_2-t_1} c_1 + \frac{t_2-t_1}{t_2-t_1} c_2 = c_2$

$$q_{0,2}(t_2) = \frac{t_2 - t_2}{t_2 - t_0} q_{0,1}(t_2) + \frac{t_2 - t_0}{t_2 - t_0} q_{1,1}(t_2) = c_2$$



(a) t = (0, 1.5, 2) (b) t = (0, 0.25, 2)

Figure 1.1 Some examples of quadratic

Note that the interpolation points are the same in plots (a) and (b). In this case of three points, the result of quadratic interpolation is clearly highly dependent on the choice of parameters. In plot (a) the value of $t_1$is being height. Thus, the traveling from c_0 to $c_1$takes more time thanc_1 to c_2. In plot (b) the value of $t_1$has been lowered leaving more 'time' for traveling from c_1 to c_2 than from c_0 to c_1. This makes the journey between these points longer and someone traveling along the curve can therefore spend the extra time allocated to this part of the journey.

3.3 General Polynomial Interpolations

The two quadratic interpolants are

$q_{0,2}(t) = q(t|c_0, c_1, c_2; t_0, t_1, t_2)$,

$$q_{1,2}(t) = q(t|c_1, c_2, c_3; t_1, t_2, t_3)$$

when the given points are $(c_i)_{i=0}^{3}$ and the choosing four parameters $t = (t_i)_{i=0}^{3}$.

Then, the cubic interpolant $q_{0,3}(t)$ is

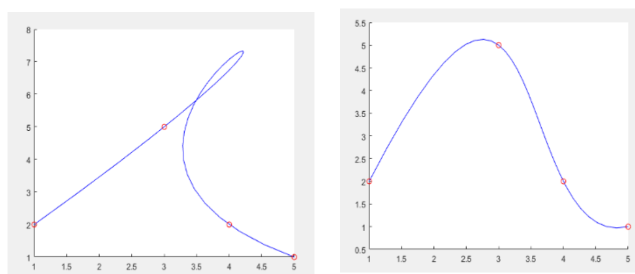
$$q_{0,3}(t) = \frac{t_3 - t}{t_3 - t_0} q_{0,2}(t) + \frac{t - t_0}{t_3 - t_0} q_{1,2}(t)$$

At$t = t_0$, $q_{0,2}(t_0) = c_0$

At$t = t_3$, $q_{1,2}(t_3) = c_3$

At an interiort_i it is a convex combination of$q_{0,1}(t_i)$ and$q_{1,1}(t_i)$ which both interpolatec_i att_i . Hence, $q_{0,3}(t_i) = c_i$ for $i = 1$ and $i = 2$ so $q_{0,3}$ interpolates the four points$(c_i)_{i=0}^{3}$.

Some examples of cubic interpolants are shown in Figure 1.2, and the same interpolation points are used in (a) and (b).



(a) t = (0, 0.35, 2.75, 3) (b) t = (0, 1.1, 2.25, 3)

Figure 1.2 Some examples of cubic interpolation

By adjusting the parameters, quite strange behavior can occur, even with these 'nice' interpolation points. In Figure (a), there is so much time to 'waste' betweenc_1 and $c_2$that the curve makes a complete loop.

In general, given $d + 1$ points$(c_i)_{i=0}^{d}$ and parameters$(t_i)_{i=0}^{d}$, the curve $q_{0,d}$of degree$d$ that satisfies $q_{0,d}(t_j) = c_j$for $j = 0,1, \ldots, d$ is constructed by forming a convex combination between the two curves of degree $d - 1$ that interpolate$(c_i)_{i=0}^{d-1}$ and$(c_i)_{i=1}^{d}$,

$$q_{0,d}(t) = \frac{t_d - t}{t_d - t_0} q_{0,d-1}(t) + \frac{t - t_0}{t_d - t_0} q_{1,d-1}(t).$$

Thus, $q_{0,d}(t)$can be written

$$q_{0,d}(t) = c_0 \ell_{0,d}(t) + c_1 \ell_{1,d}(t) + \cdots + c_d \ell_{d,d}(t),$$

where the functions$\{\ell_{i,d}\}_{i=0}^{d}$ are the Lagrange polynomials of degree$d$ given by

$$\ell_{i,d}(t) = \prod_{\substack{0\le j\le d \\ j\neq 1}} \frac{(t-t_j)}{t_i - t_j}$$

These polynomials satisfy the condition

$$\ell_{i,d}(t_k) = \begin{cases} 1\ , & \text{if } k = i \\ 0\ , & \text{otherwise} \end{cases}$$

which is necessary since $q_{0,d}(t_k) = c_k$.

4. Algorithm (Neville/Aitken Method)

Let d be a positive integer and let the $d+1$ points $(c_i)_{i=0}^{d}$ be given together with $d+1$ strictly increasing parameter values $t = (t_i)_{i=0}^{d}$. There is a polynomial curve $q_{0,d}$ of degree d that satisfies the conditions

$$q_{0,d}(t_i) = c_i \qquad \text{for } i = 0,1,\dots,d,$$

and for any real number t the following algorithm computes the points $q_{0,d}(t)$. First set $q_{i,0}(t) = c_i$ for $i = 0,1,\dots,d$, and then compute

$$q_{i,r}(t) = \frac{t_{i+r}-t}{t_{i+r}-t_i} q_{i,r-1}(t) + \frac{t-t_i}{t_{i+r}-t_i} q_{i+r,r-1}(t)$$

for $i = 0,1,\dots,d-r$, and $i = 1,2,\dots,d$,

The computations involved in determining a cubic interpolation curve are shown in the triangular table in Figure1.3.

The computations starts from the right and proceed to the left and at any point a quantity $q_{i,r}$ is computed by combining, in an affine combination, the two quantities at the beginning of the two arrows meeting at $q_{i,r}$. The expression between the two arrows is the denominator of the weights in the affine combination while the two numerators are written along the respective arrows.

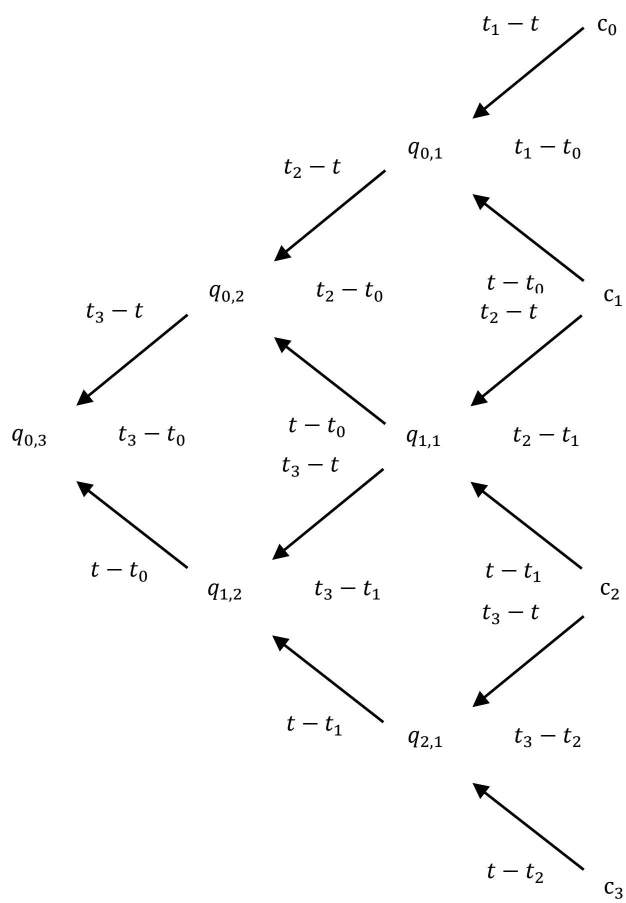
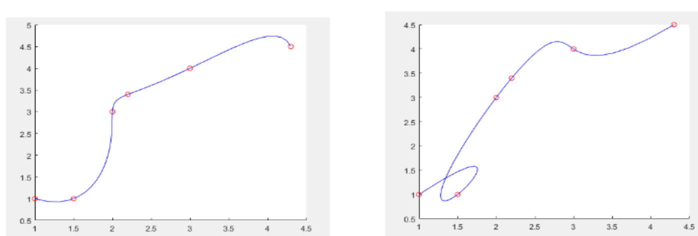


Figure 1.3.Computing a point on a cubic interpolating curve

Two examples of curves of degree five are shown in Figure 1.4, both interpolating same points.



(a) t = (0, 0.35, 1.9, 2.8, 3.75, 5) (a) t = (0, 1.1, 2.8, 3, 4.2, 5)

Figure 1.4 Two examples of Interpolation withpolynomial curves of degree five

5. CONCLUSION

We had presented polynomial interpolation and cubic interpolating curves that are fundamental background for “Spline interpolation”. Spline interpolation is applied to design the smoothness of shape of automobile, ships, and many others.

6. ACKNOWLEDGEMENTS

First, the authors would like to acknowledge the support of the papers for their references. The authors would like to acknowledgement our parents and all our teachers throughout our studies.

7. REFERENCES

[1]B.P.Demidovish, I.A Maron, 1973, "Computational Mathematics" , Mir Publisher, Moscow.

[2] Brian R.Hunt, Ronald L.Lipsman, Jonathan M. Rosenberg, "A Guide to MATLAB for Beginners and Experienced Users" , Cambridge University Press.

[3] Keryszig.E, 1993," Advanced Engineering Mathematics", Eight Edition, John Wiley and Sons, Inc. New York.

[4] Tom Lyche and Knut Morken, 2002," Spline Methods Draft'', University of Oslo.

8. APPENDICES

Using the MATLAB, the interpolating curves are easily obtained.

Figure 1.1

```
Inter_curve=[];
t0=0;t1=1.5;t2=2;c0=[1 1];c1=[2 2];c2=[4 3];
for t=0:0.1:2;
   q01=(t1-t)/(t1-t0)*c0+(t-t0)/(t1-t0)*c1;
   q11=(t2-t)/(t2-t1)*c1+(t-t1)/(t2-t1)*c2;
   q02=(t2-t)/(t2-t0)*q01+(t-t0)/(t2-t0)*q11;
Inter_curve=[Inter_curve;q02];
   axis on;
   hold on;
plot(Inter_curve(:,1),Inter_curve(:,2),'b',[1 2 4],[1 2 3],'ro')
end
Inter_curve=[];
t0=0;t1=0.25;t2=2;c0=[1 1];c1=[2 2];c2=[4 3];
for t=0:0.1:2;
   q01=(t1-t)/(t1-t0)*c0+(t-t0)/(t1-t0)*c1;
   q11=(t2-t)/(t2-t1)*c1+(t-t1)/(t2-t1)*c2;
   q02=(t2-t)/(t2-t0)*q01+(t-t0)/(t2-t0)*q11;
Inter_curve=[Inter_curve;q02];
   axis on;
   hold on;
plot(Inter_curve(:,1),Inter_curve(:,2),'b',[1 2 4],[1 2 3],'ro')
end
```

Figure 1.2

```
Inter_curve=[];
t0=0;t1=0.35;t2=2.75;t3=3;c0=[1 2];c1=[3 5];c2=[4 2];c3=[5 1];
for t=0:0.1:3;
   q01=(t1-t)/(t1-t0)*c0+(t-t0)/(t1-t0)*c1;
   q11=(t2-t)/(t2-t1)*c1+(t-t1)/(t2-t1)*c2;
   q02=(t2-t)/(t2-t0)*q01+(t-t0)/(t2-t0)*q11;
   q21=(t3-t)/(t3-t2)*c2+(t-t2)/(t3-t2)*c3;
   q12=(t3-t)/(t3-t1)*q11+(t-t1)/(t3-t1)*q21;
   q03=(t3-t)/(t3-t0)*q02+(t-t0)/(t3-t0)*q12;
Inter_curve=[Inter_curve;q03];
   axis on;
   hold on;
plot(Inter_curve(:,1),Inter_curve(:,2),'b',[1 3 4 5],[2 5 2 1],'ro')
end

Inter_curve=[];
t0=0;t1=1.1;t2=2.25;t3=3;c0=[1 2];c1=[3 5];c2=[4 2];c3=[5 1];
for t=0:0.1:3;
   q01=(t1-t)/(t1-t0)*c0+(t-t0)/(t1-t0)*c1;
   q11=(t2-t)/(t2-t1)*c1+(t-t1)/(t2-t1)*c2;
   q02=(t2-t)/(t2-t0)*q01+(t-t0)/(t2-t0)*q11;
   q21=(t3-t)/(t3-t2)*c2+(t-t2)/(t3-t2)*c3;
   q12=(t3-t)/(t3-t1)*q11+(t-t1)/(t3-t1)*q21;
   q03=(t3-t)/(t3-t0)*q02+(t-t0)/(t3-t0)*q12;
Inter_curve=[Inter_curve;q03];
   axis on;
   hold on;
plot(Inter_curve(:,1),Inter_curve(:,2),'b',[1 3 4 5],[2 5 2 1],'ro')
end
```

Figure 1.4

```
Inter_curve=[];
t0=0;t1=0.35;t2=1.9;t3=2.8;t4=3.75;t5=5;c0=[1 1];c1=[1.5
1];c2=[2 3];c3=[2.2 3.4];c4=[3 4];c5=[4.3 4.5];
for t=0:0.1:5;
   q01=(t1-t)/(t1-t0)*c0+(t-t0)/(t1-t0)*c1;
   q11=(t2-t)/(t2-t1)*c1+(t-t1)/(t2-t1)*c2;
   q02=(t2-t)/(t2-t0)*q01+(t-t0)/(t2-t0)*q11;
   q21=(t3-t)/(t3-t2)*c2+(t-t2)/(t3-t2)*c3;
   q12=(t3-t)/(t3-t1)*q11+(t-t1)/(t3-t1)*q21;
   q03=(t3-t)/(t3-t0)*q02+(t-t0)/(t3-t0)*q12;
   q31=(t4-t)/(t4-t3)*c3+(t-t3)/(t4-t3)*c4;
   q41=(t5-t)/(t5-t4)*c4+(t-t4)/(t5-t4)*c5;
   q22=(t4-t)/(t4-t2)*q21+(t-t2)/(t4-t2)*q31;
   q13=(t4-t)/(t4-t1)*q12+(t-t1)/(t4-t1)*q22;
   q32=(t5-t)/(t5-t3)*q31+(t-t3)/(t5-t3)*q41;
   q23=(t5-t)/(t5-t2)*q22+(t-t2)/(t5-t2)*q32;
   q14=(t5-t)/(t5-t1)*q13+(t-t1)/(t5-t1)*q23;
   q04=(t4-t)/(t4-t0)*q03+(t-t0)/(t4-t0)*q13;
   q05=(t5-t)/(t5-t0)*q04+(t-t0)/(t5-t0)*q14;
Inter_curve=[Inter_curve;q05];
   axis on;
   hold on;
plot(Inter_curve(:,1),Inter_curve(:,2),'b',[1 1.5 2 2.2 3 4.3],[1 1 3
3.4 4 4.5],'ro')
end


Inter_curve=[];
t0=0;t1=1.1;t2=2.8;t3=3;t4=4.2;t5=5;c0=[1 1];c1=[1.5 1];c2=[2
3];c3=[2.2 3.4];c4=[3 4];c5=[4.3 4.5];
for t=0:0.1:5;
   q01=(t1-t)/(t1-t0)*c0+(t-t0)/(t1-t0)*c1;
   q11=(t2-t)/(t2-t1)*c1+(t-t1)/(t2-t1)*c2;
   q02=(t2-t)/(t2-t0)*q01+(t-t0)/(t2-t0)*q11;
   q21=(t3-t)/(t3-t2)*c2+(t-t2)/(t3-t2)*c3;
   q12=(t3-t)/(t3-t1)*q11+(t-t1)/(t3-t1)*q21;
   q03=(t3-t)/(t3-t0)*q02+(t-t0)/(t3-t0)*q12;
   q31=(t4-t)/(t4-t3)*c3+(t-t3)/(t4-t3)*c4;
   q41=(t5-t)/(t5-t4)*c4+(t-t4)/(t5-t4)*c5;
   q22=(t4-t)/(t4-t2)*q21+(t-t2)/(t4-t2)*q31;
   q13=(t4-t)/(t4-t1)*q12+(t-t1)/(t4-t1)*q22;
```

```
   q32=(t5-t)/(t5-t3)*q31+(t-t3)/(t5-t3)*q41;
   q23=(t5-t)/(t5-t2)*q22+(t-t2)/(t5-t2)*q32;
   q14=(t5-t)/(t5-t1)*q13+(t-t1)/(t5-t1)*q23;
   q04=(t4-t)/(t4-t0)*q03+(t-t0)/(t4-t0)*q13;
   q05=(t5-t)/(t5-t0)*q04+(t-t0)/(t5-t0)*q14;
Inter_curve=[Inter_curve;q05];
   axis on;
   hold on;
plot(Inter_curve(:,1),Inter_curve(:,2),'b',[1 1.5 2 2.2 3 4.3],[1 1 3
3.4 4 4.5],'ro')
end
```